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# ONE-DIMENSIONAL SHOCK WAVES IN INHOMOGENEOUS ELASTIC MATERIALS

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## 1. INTRODUCTION

IN THIS paper, we derive a differential equation which governs the behavior of the amplitudes of shock waves propagating in inhomogeneous elastic materials without assuming that the regions ahead of the waves are at rest. We show that there exists a number $\lambda$, called the critical jump in strain gradient, and that the behavior of the amplitude of a shock depends on the relative magnitudes of $\lambda$ and the jump in strain gradient across the shock. This critical jump in strain gradient depends on the local elastic properties of a given material, the material inhomogeneity and the nature of the strain field ahead of the shock.

## 2. CONSTITUTIVE ASSUMPTIONS AND GENERAL PROPERTIES OF SHOCK WAVES

For an inhomogeneous elastic material,† the value of the stress $T(X, t)$ at the material point $X$ and time $t$ depends on the value of the strain $\varepsilon = \varepsilon(X, t)$ and the material point $X$:

$$T(X,t) = \hat{T}(\varepsilon; X). \tag{2.1}$$

Of course, the strain $\varepsilon$ is given by $\varepsilon(X, t) = \partial u(X, t)/\partial X$, where $u = u(X, t)$ is the displacement at time $t$ of the material point $X$. In other words, there is no reference configuration which renders the response function $\hat{T}(\cdot\,;\cdot)$ independent of $X$; hence the density $\rho_R$, in the chosen configuration, also depends on $X$, i.e. $\rho_R = \hat{\rho}_R(X)$.

We assume that $\hat{T}(\cdot\,; X)$ is of class $C^2$. The quantities

$$E = \frac{\partial \hat{T}}{\partial \varepsilon}, \qquad \tilde{E} = \frac{\partial^2 \hat{T}}{\partial \varepsilon^2} \tag{2.2}$$

are called the *tangent modulus* and the *second-order tangent modulus*, respectively. We further assume that $\hat{T}(\cdot\,;\cdot)$, $\partial\hat{T}/\partial X(\cdot\,,\cdot)$, $E(\cdot\,;\cdot)$ and $\tilde{E}(\cdot\,;\cdot)$ are of class $C^0$, $E(\varepsilon;\cdot)$ and $\hat{\rho}_R(\cdot)$ are differentiable and

$$E(\varepsilon; X) > 0, \qquad \tilde{E}(\varepsilon; X) \neq 0. \tag{2.3}$$

† We suspect that this theory may be applied to describe the gross behavior of waves in certain classes of composite materials. See, for example, Barker [1] who calculated numerically the wave profiles in laminates of alternating layers of two elastic materials. His solutions exhibit that shock waves are possible. Further, he also has solutions exhibiting the existence of acceleration waves.

We assume that the motion of the body contains a shock wave moving with intrinsic velocity

$$U(t) = \frac{\mathrm{d}Y(t)}{\mathrm{d}t} > 0, \tag{2.4}$$

where $Y(t)$ is the material point at which the wave is to be found at time $t$. That is, $u(\cdot,\cdot)$ is a continuous function everywhere; while $\varepsilon(\cdot,\cdot)$ and $\dot{u}(\cdot,\cdot)$ have jump discontinuities across the shock wave, they are continuous functions everywhere else.

The compatibility relation†

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = [\dot{f}] + U\left[\frac{\partial f}{\partial X}\right] \tag{2.5}$$

with $f(\cdot,\cdot) = u(\cdot,\cdot)$ implies that

$$U[\varepsilon] = -[\dot{u}]. \tag{2.6}$$

Here, for a function $f(\cdot,\cdot)$, $[f] = f^- - f^+$, with $f^\pm = \lim_{X\to Y^\pm(t)} f(X,t)$.‡ The jump $[\varepsilon]$ in strain is called the *amplitude* of the shock. A wave is said to be a compression shock if $[\varepsilon] < 0$; a wave is said to be an expansion shock if $[\varepsilon] > 0$. This is motivated by the fact that $[\varepsilon] < 0 \Leftrightarrow [\rho] > 0$ and $[\varepsilon] > 0 \Leftrightarrow [\rho] < 0$, where $\rho$ is the present density of the material.§ In other words, across a compression shock the density $\rho^-$ behind the wave is greater than the density $\rho^+$ ahead of the wave; and, across an expansion shock the density $\rho^-$ behind the wave is less than the density $\rho^+$ ahead of the wave. In particular, if the material region ahead of the wave is at rest and unstrained, so that $\varepsilon^+ = 0$, then a wave is a compression shock or an expansion shock according as $\varepsilon^-$ is negative or positive.

Balance of momentum asserts that

$$[T] = -\rho_R U[\dot{u}] \tag{2.7}$$

across the shock, and away from the shock‖

$$\frac{\partial T}{\partial X} = \rho_R \ddot{u}. \tag{2.8}$$

By (2.6) and (2.7), the intrinsic velocity of the shock is given by the well-known formula

$$U^2 = \frac{[T]}{\rho_R[\varepsilon]}. \tag{2.9}$$

Further, it follows from (2.6), (2.8) and the compatibility relation (2.5) with $f(\cdot,\cdot) = \dot{u}(\cdot,\cdot)$ and $\varepsilon(\cdot,\cdot)$ that

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + [\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = U^2\left[\frac{\partial \varepsilon}{\partial X}\right] - \frac{1}{\rho_R}\left[\frac{\partial T}{\partial X}\right] \tag{2.10}$$

which the amplitude of the shock must obey.¶ Formula (2.10) is a consequence of balance of momentum and the kinematical conditions for a shock; it does not depend on the constitutive relation for the stress.

† See, for example, Thomas [2], Coleman and Gurtin [3]. A rigorous derivation of the compatibility relation is given by Chen and Wicke [4].

‡ That is, $f^+$ and $f^-$ are the limiting value of $f(\cdot,\cdot)$ immediately in front of and behind the wave.

§ See, for example, Chen [5, Section 5] for the details of deviation of these assertions.

‖ It is assumed that there is no external body force.

¶ Formula (2.10) is given independently by Achenbach and Herrmann [6] and Chen and Gurtin [7].

## 3. THE SHOCK AMPLITUDE EQUATION FOR WAVES IN INHOMOGENEOUS MATERIALS

It follows immediately from the constitutive relation (2.1) and ($2.2_1$) that away from the shock

$$\frac{\partial T}{\partial X} = E\frac{\partial \varepsilon}{\partial X}+\frac{\partial \hat{T}}{\partial X},$$

which together with (2.10) imply that

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}+[\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = \left(U^2-\frac{E^-}{\rho_R}\right)\left[\frac{\partial \varepsilon}{\partial X}\right]-\frac{[E]}{\rho_R}\left(\frac{\partial \varepsilon}{\partial X}\right)^+ -\frac{1}{\rho_R}\left[\frac{\partial \hat{T}}{\partial X}\right] \tag{3.1}$$

where

$$E^- = E(\varepsilon^-, Y(t)), \qquad E^+ = E(\varepsilon^+, Y(t)). \tag{3.2}$$

In order that we may simplify (3.1) we need an expression for $\mathrm{d}U/\mathrm{d}t$. Thus, differentiating (2.9) and utilizing (2.1) and ($2.2_1$) we have

$$2\rho_R U\frac{\mathrm{d}U}{\mathrm{d}t} = -U^2\frac{\mathrm{d}\rho_R}{\mathrm{d}t}+\frac{E^-}{[\varepsilon]}\frac{\mathrm{d}\varepsilon^-}{\mathrm{d}t}+\frac{U}{[\varepsilon]}\left(\frac{\partial \hat{T}}{\partial X}\right)^- -\frac{E^+}{[\varepsilon]}\frac{\mathrm{d}\varepsilon^+}{\mathrm{d}t}-\frac{U}{[\varepsilon]}\left(\frac{\partial \hat{T}}{\partial X}\right)^+ -\frac{\rho_R U^2}{[\varepsilon]}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t},$$

which may be written in the form

$$2\rho_R U\frac{\mathrm{d}U}{\mathrm{d}t} = -U^2\frac{\mathrm{d}\rho_R}{\mathrm{d}t}+\frac{(E^- -\rho_R U^2)}{[\varepsilon]}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}+\frac{[E]}{[\varepsilon]}\frac{\mathrm{d}\varepsilon^+}{\mathrm{d}t}-\frac{U}{[\varepsilon]}\left[\frac{\partial \hat{T}}{\partial X}\right], \tag{3.3}$$

where $\mathrm{d}\varepsilon^+/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}\varepsilon^+}{\mathrm{d}t} = \dot{\varepsilon}^+ + U\left(\frac{\partial \varepsilon}{\partial X}\right)^+. \tag{3.4}$$

Thus, by (3.1), (3.3) and (3.4) we have the following:

*The amplitude of a shock wave propagating in an inhomogeneous elastic material obeys the equation*

$$\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = \frac{E^- -\rho_R U^2}{2\rho_R U[1+(E^- -\rho_R U^2)/(4\rho_R U^2)]}\left\{\lambda-\left[\frac{\partial \varepsilon}{\partial X}\right]\right\}, \tag{3.5}$$

*where*

$$\lambda = -\frac{1}{2(E^- -\rho_R U^2)}\left\{[E]\left\{\frac{\dot{\varepsilon}^+}{U}+3\left(\frac{\partial \varepsilon}{\partial X}\right)^+\right\}-\left[\frac{\partial \hat{T}}{\partial X}\right]+[\varepsilon]U\frac{\mathrm{d}\rho_R}{\mathrm{d}t}\right\}. \tag{3.6}$$

Formula (3.5) is quite complicated; in general it is not possible to deduce any information from it without adopting additional assumptions. In the following section, we shall consider certain specific applications of (3.5) for which we can deduce definite conclusions.

## 4. THE BEHAVIOR OF CERTAIN COMPRESSION SHOCKS AND EXPANSION SHOCKS

Here, we shall consider certain specific applications of (3.5). In particular, we shall consider

(a) *a* $\begin{pmatrix}\textit{compression}\\ \textit{expansion}\end{pmatrix}$ *shock entering a material which is initially in* $\begin{pmatrix}\textit{compression}\\ \textit{tension}\end{pmatrix}$, *and*

(b) *a* $\begin{pmatrix}\textit{compression}\\ \textit{expansion}\end{pmatrix}$ *shock entering a material which is initially in* $\begin{pmatrix}\textit{tension}\\ \textit{compression}\end{pmatrix}$ *such that the material behind the wave remains in* $\begin{pmatrix}\textit{tension}\\ \textit{compression}\end{pmatrix}$.

*Case* (a)

Here, in considering a compression shock, for which

$$\varepsilon^+ < 0 \quad \text{and} \quad [\varepsilon] < 0 \tag{4.1}$$

we assume that the local stress–strain law in compression is concave from below/ i.e.

$$\tilde{E}(\varepsilon; X) < 0 \qquad \text{for } \varepsilon \leq 0; \tag{4.2}$$

in considering an expansion shock, for which

$$\varepsilon^+ > 0 \quad \text{and} \quad [\varepsilon] > 0, \tag{4.3}$$

we assume that the local stress–strain law in tension is concave from above; i.e.

$$\tilde{E}(\varepsilon; X) > 0 \qquad \text{for } \varepsilon \geq 0. \tag{4.4}$$

The assumptions (4.2) and (4.4) are not artificial. In fact they are the necessary conditions for the existence of the compression shock and the expansion shock; they are also consistent with the conditions under which the amplitudes of compressive and expansive acceleration waves can become infinite.† In either case (4.1) and (4.2) or (4.3) and (4.4) with $(2.3_1)$, (2.9) and (3.2) imply that

$$E^- > \rho_R U^2, \tag{4.5}$$

and

$$[E] > 0. \tag{4.6}$$

Thus, by (3.5) and (4.5), we have the following:

(i) *Consider a compression shock entering a material which is initially in compression. Suppose that the local stress–strain law in compression is concave from below. Then at any instant*

$$\left[\frac{\partial \varepsilon}{\partial X}\right] < \lambda \Leftrightarrow \frac{\mathrm{d}\|[\varepsilon]\|}{\mathrm{d}t} < 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] > \lambda \Leftrightarrow \frac{\mathrm{d}\|[\varepsilon]\|}{\mathrm{d}t} > 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] = \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = 0.$$

† Compare Coleman *et al.* [8]. In this regard, also refer to Bailey and Chen [9, Theorem 4.3].

(ii) *Consider an expansion shock entering a material which is initially in tension. Suppose that the local stress–strain law in tension is concave from above. Then at any instant*

$$\left[\frac{\partial \varepsilon}{\partial X}\right] < \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} > 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] > \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} < 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] = \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = 0.$$

*Case* (b)

Here, in considering a compression shock, for which

$$\varepsilon^- > 0, \qquad \varepsilon^+ > 0 \quad \text{and} \quad [\varepsilon] < 0, \tag{4.7}$$

we assume that the local stress–strain law in tension is convex from above; i.e.

$$\tilde{E}(\varepsilon; X) < 0 \qquad \text{for } \varepsilon \geq 0; \tag{4.8}$$

in considering an expansion shock, for which

$$\varepsilon^- < 0, \qquad \varepsilon^+ < 0 \quad \text{and} \quad [\varepsilon] > 0, \tag{4.9}$$

we assume that the local stress–strain law in compression is convex from below; i.e.

$$\tilde{E}(\varepsilon, X) > 0 \qquad \text{for } \varepsilon \leq 0. \tag{4.10}$$

Notice that these conditions are in contrast to those for Case (a); of course, (4.8) and (4.10) are the necessary conditions for the existence of the compression shock and the expansion shock. Here again, we can show that in either case

$$E^- - \rho_R U^2 > 0, \qquad [E] > 0. \tag{4.11}$$

Thus, by (3.5) and ($4.11_1$), we have the following:

(i) *Consider a compression shock entering a material which is initially in tension such that the material behind the wave remains in tension. Suppose that the local stress–strain law in tension is convex from above. Then at any instant*

$$\left[\frac{\partial \varepsilon}{\partial X}\right] < \lambda \Leftrightarrow \frac{\mathrm{d}|[\varepsilon]|}{\mathrm{d}t} < 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] > \lambda \Leftrightarrow \frac{\mathrm{d}|[\varepsilon]|}{\mathrm{d}t} > 0,$$

$$\left[\frac{\partial \varepsilon}{\partial X}\right] = \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = 0.$$

(ii) *Consider an expansion shock entering a material which is initially in compression such that the material behind the wave remains in compression. Suppose that the local stress–strain law in compression is convex from below. Then at any instant*

$$\left[\frac{\partial\varepsilon}{\partial X}\right] < \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} > 0,$$

$$\left[\frac{\partial\varepsilon}{\partial X}\right] > \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} < 0,$$

$$\left[\frac{\partial\varepsilon}{\partial X}\right] = \lambda \Leftrightarrow \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = 0.$$

In view of the fact that the criteria governing the growth and decay of the amplitudes of shock waves are based on the relative magnitudes of the jump in strain gradient $[\partial\varepsilon/\partial X]$ and $\lambda$, we call the number $\lambda$, defined in (3.6), the *critical jump in strain gradient* for shock waves in inhomogeneous elastic materials. Notice that, for a given material, $\lambda$ depends on its local elastic properties as well as the material inhomogeneity; it also depends on the properties of the motion ahead of the shock. Notice also that the results for Cases (a) and (b) are the same in consequence.

Finally, we should also point out that even if the material ahead of the wave is at rest in its reference configuration, the reduced form of the critical jump in strain gradient $\lambda$, defined by (3.6), will not vanish. Thus, in view of an earlier study of the behavior of shock waves in non-linear viscoelastic materials by Chen and Gurtin [7], we have the following important observation:

*The behavior of the amplitude of a shock wave propagating in an inhomogeneous elastic material at rest in its reference configuration is qualitatively the same as that of a wave in a non-linear viscoelastic material which is initially unstrained.*

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